One-Dimensional Equations Governing Single-Cavity Die Design

Steven J. Weinstein and Kenneth J. Ruschak
Emulsion Coating Technologies, Eastman Kodak Company, Rochester, NY 14652

The 1-D approximate equations governing flow in a single-cavity die have been examined in detail. Although the 1-D momentum equation in the slot and mass conservation equation used in previous studies are generally the same, various forms of the 1-D momentum equation governing flow in the cavity have been utilized. A rigorous accounting of the origin of the 1-D cavity equation is given, via asymptotic techniques, integral equations, and the use of approximate velocity fields. Additionally, these techniques are employed to place the slot flow and mass conservation equations, utilized in previous studies, on a firmer theoretical ground. The derived cavity equation and differential equation system is found to be identical to that of Leonard (1985).

Introduction

Extrusion dies are often used in manufacture of polymeric sheets and photographic products. In such operations, the die is used to form a two-dimensional liquid film that is wide compared with its thickness. The simplest possible die consists of a single distribution chamber (referred to as a cavity) and a slot, both of width W, from which the liquid film emerges (Figure 1). The resistance to flow in the cavity is made low by choosing a relatively large cross-sectional area, A, while the slot is designed such that its resistance to flow is high; this is accomplished by choosing a small slot height, h, and a relatively long slot length, L. In this way, fluid entering the distribution chamber tends to distribute widthwise (the z-direction in Figure 1a) before entering the slot, in which the flow is oriented primarily toward the slot exit (i.e., the x-direction in Figure 1a). Equivalently, the idea in die design is to try to obtain a constant cavity pressure, so that, if the fluid exiting the die experiences a constant pressure (such as atmospheric), the flow in the slot will be widthwise uniform.

In a die having constant cavity and slot dimensions, the liquid film emerging from a die is not precisely uniform, owing in part to the fact that a constant cavity pressure cannot be obtained with a finite-sized cavity. Clearly, there must be some pressure drop across the cavity width W to force the fluid to distribute widthwise in the first place. Adjustments in the die geometry are often employed to compensate for the pressure variation in the cavity, such as widthwise varying slot lengths or cavity areas (as shown in Figure 1a); in some cases, a second die cavity is added (such dies are called dual-cavity

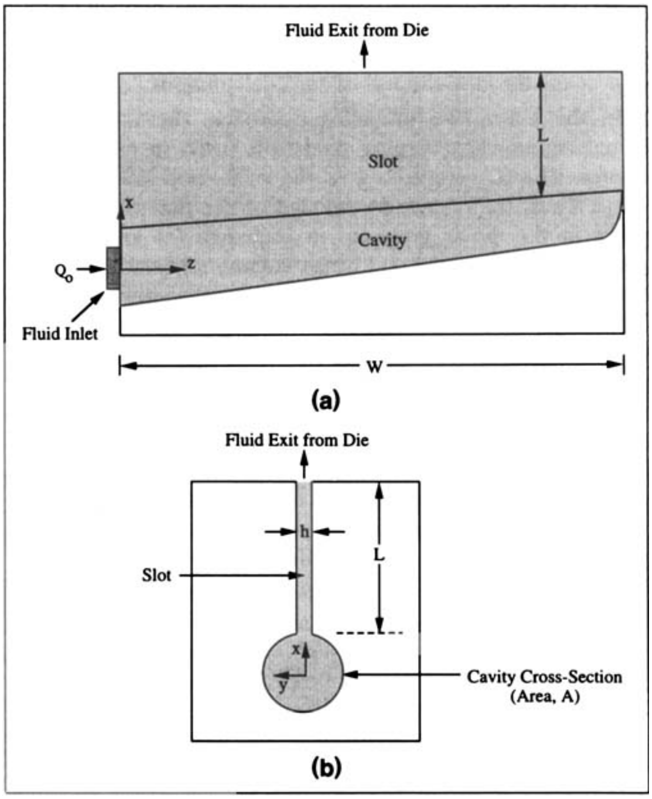

Figure 1. Typical geometry of a single-cavity die: (a) widthwise view; (b) side view.

designs). Theoretical models of die flows are an invaluable tool in identifying these geometrical adjustments, as well as in predicting the performance of a given die for ranges of fluid and flow conditions.

There are two principal theoretical techniques used to analyze widthwise flow nonuniformities in a coating die: three-dimensional computation and modeling approaches. When three-dimensional computation is utilized, the full-blown equations governing fluid flow are solved to obtain pointwise velocity and pressure fields in the cavities and slots (see, for example, Nguyen and Kamal, 1990; Wang, 1991a,b; Wen and Liu, 1994). Because these equations are well known, commercially available software packages can be used to generate solutions to the equations, and so numerical codes do not necessarily need to be written from scratch. However, these governing equations are nonlinear, which means that iterative, time-consuming solution algorithms are used in such packages, and there is no guarantee that such algorithms will converge to a solution. Furthermore, the disparate geometrical and flow characteristics in the cavity and slots, discussed before, are often difficult to handle numerically. To adequately resolve widely disparate geometries and flows using, say, a finite-element computational method, one needs to choose a computational mesh that adequately captures not only the slot and cavity regions, but also captures the rapid change in flow characteristics as the fluid moves from one region to the other. For the case of die flows, fine-mesh resolution is necessary to pick up this behavior, which greatly increases computational times. Thus, although quite accurate, three-dimensional computation is numerically intensive, often requiring hours of computer time to accurately simulate a single die flow condition. For this reason, the number of flow conditions and die geometries that can be assessed by computation is quite limited, and it is difficult to optimize a process solely through the use of such calculations.

In the alternative modeling approach, the three-dimensional equations governing fluid flow (used in the previous approach) are averaged across the cavity and slot cross sections. Thus, the precise knowledge of the flow field at each point in the die is given up in exchange for average flow properties such as the widthwise volumetric flow rate in the cavity, the cavity pressure, and the volumetric flow per width in the slot. The advantage of the averaged equations is that they are much easier to solve, and many flow conditions and die geometries can be investigated quite quickly. The disadvantage of the modeling approach is that the governing equations are approximate, and thus there is some error incurred in their use. Note, however, that since the equations used in modeling are derivable from the full three-dimensional equations, the sources of such errors, and to some extent the magnitude of such errors, are quantifiable.

Based on the preceding comments, the best theoretical approach to die design, then, utilizes both modeling and computation. The modeling approach can be used to determine feasible designs, for ranges of flows and fluid rheologies, in a quick and efficient manner. From these designs, the most promising ones can be analyzed by the more precise and intensive three-dimensional computation, and adjustments to such designs, necessary to correct for approximation errors in the modeling approach, can be made. Such a combined use of modeling and three-dimensional computation is advantageous in that it can reduce the time to design a die. Ultimately, the final designs obtained from such a theoretical procedure must be examined experimentally, since experiments are the ultimate test and justification for any of the theoretical approaches used. In this report, attention is focused on the modeling approach.

Using a modeling approach to examine single- and dual-cavity die flows is by no means new, and there is extensive academic and trade literature on the subject; we will make no attempt to review all of it here. Much of the work on single-cavity dies has focused on the "viscous-dominated" regime in both the cavity and slot, and for power-law fluids. Viscous-dominated analyses have in common the assumption that the cavity flow rate, and hence pressure gradient, slowly varies in the z-direction (Figure 1a) as fluid "leaks" into the slot. Despite the depletion of flow in the cavity, the pressure drop–flow rate relationship is assumed to have a fully developed form, such as the Poiseuille relationship for Newtonian flow in a pipe. In the slot, the pressure drop and flow relationship is assumed to be that of unidirectional flow between parallel walls (in the x-direction of Figure 1), despite the fact that the average slot velocity can vary due to pressure variations in the z-direction. Thus, viscous-dominated analyses are one-dimensional, in that the equations governing the flow in the die are solely z-dependent (Figure 1a).

In initial viscous-dominated work examining single-cavity dies, power-law fluids flowing in relatively simple geometries having circular cavities and straight slots were examined (Carley, 1954; McKelvey and Ito, 1971, among many others). Such viscous dominated analyses have been broadened by many to encompass geometries in which the cavity shape, cross-sectional area, slot lengths, and slot height can vary (see, for example, Pearson, 1964; Liu et al., 1988). In these more complicated geometries, the general form of the pressure drop vs. flow relationship in the slot is not altered from the unidirectional relationship associated with constant slot dimensions. Additionally, the specific effects of cavity shape on flow are incorporated into the viscous term via a viscous shape factor (Miller, 1972; Liu, 1983), which is constant for a given cross-sectional shape and power-law index (related to the slope of the viscosity-strain rate curve in log-log space; see Bird et al., 1977), even if the cross-sectional area varies (see Liu et al., 1988).

More recently, Leonard (1985) has generalized the viscous-dominated model to include inertial and gravitational effects in the cavity. In his cavity equation, the previously used viscous model is obtained when inertial and gravitational effects are neglected. The equation relating the pressure drop and flow across the slot is identical to that used with the viscous model. To derive his cavity equation, Leonard utilizes the approach of Lundgren et al. (1964) and Huang and Yu (1973), who respectively consider entrance region and porous media flows in ducts of constant cross-sectional areas. Since the dominant flow in the cavity is in the z-direction in Figure 1a (in keeping with the "leaky-pipe" ideas cited previously), Leonard (1985) starts with the full-blown z-component of the equation of motion, neglecting only the viscous stresses in the z-direction. The assumption here is that viscous stresses across the cavity cross section will be large compared with those in the direction of motion, which is likely to be valid provided the cavity is long compared with a characteristic cross-sectional length (Huang and Yu, 1973), that is, $A^{1/2}$. Despite the fact that he focuses on the z-component in the equation of motion, Leonard does not assume that the x and y velocity components, lying in the cavity cross section (Figure 1b), are zero; for it is the presence of these components which allows for the slow leakage into the slot, as well as the appearance of nonzero inertial effects. Thus, the full-blown continuity equation is utilized as well, where the x and y velocity components explicitly appear in the z-component of the momentum equation. By integrating the z-component of the momentum equation across the cavity cross-section, utilizing the continuity equation, and applying integral theorems, a single integral equation is obtained in which only the z-component of velocity explicitly appears. Then, by approximating the velocity field in the cavity as fully developed in any cavity cross section (but letting this velocity field vary in the z-direction), and substituting into the integral equation, a one-dimensional cavity equation is obtained in which the z-dependent flow rate and pressure are related. Through this velocity-field approximation, the viscous shape factor explicitly arises; furthermore, an additional shape factor arises associated with the inertial term, which is called the inertia shape parameter or kinetic shape factor (Leonard, 1985; Lee and Liu, 1989). As for the viscous shape factor, the kinetic shape factor is only a function of the cavity shape and power-law index.

Sartor (1990) obtains a different cavity equation in his analysis of single-cavity dies. He assumes that the only nonzero component of velocity in the cavity is in the z-direction, even when there is flow leakage into the slot. Furthermore, in contrast to the analysis of Leonard (1985), Sartor does not neglect the viscous stresses in the dominant direction of flow (the z-direction). Sartor also utilizes a full Hele–Shaw analysis (Schlichting, 1979) of the slot region, which implicitly assumes that slot velocity variations in the x- and z-directions of Figure 1a are coupled and of the same order; that is, Sartor's slot treatment is two-dimensional. Durst et al. (1994) utilizes a similar cavity analysis to that of Sartor, although the one-dimensional slot equation of Leonard (1985) and previous viscous analyses is used. As pointed out by Durst et al. (1994), since the purpose of a die is to obtain a widthwise uniform flow at its slot exit, it would seem that the one-dimensional slot equation would be reasonable to utilize, at least with practical designs; that is, because a die design would not be accepted for significant widthwise variation in flow (for which the x and z velocity components were of the same order). We note here that two-dimensional Hele–Shaw analyses have been used to examine die flows in which the boundaries of the slot are markedly sloped (Vrahopoulou, 1991), or in which the slot flow is obstructed by a foreign object (Vrahopoulou, 1992).

Lee and Liu (1989) use a similar equation to Leonard (1985) to model the flow in the inner distribution chamber (the inner cavity) of a dual-cavity die (the function of this inner cavity is identical to that of the cavity in the single-cavity case). However, the form of the inertial term is slightly different. Lee and Liu (1989) have an inertial term containing a derivative of the form $d(Q^2/A^2)/dz$, while the inertial term of Leonard (1985) has a derivative as $(1/A)\ d(Q^2/A)/dz$, where A is the cavity cross-sectional area, and Q is the z-dependent (widthwise) cavity volumetric flow rate. For a cavity having a constant cross-sectional area, the inertial terms of Leonard (1985) and Lee and Liu (1989) are identical; however, in general, the cavity equations are applied to cases where the cavity area varies with z as well, making the discrepancy between these equations troublesome. It is interesting to note that Lee and Liu (1989) and Leonard (1985) perform essentially the same derivation of the cavity equation in their articles. However, in both articles, the details of the final steps in approximating the inertial integral (obtained via averaging across the cavity cross section) with a one-dimensional differential form are not explicitly given; it is this final step that leads to the discrepancy between the equations. It is notable that Lee and Liu (1989) cite the equation of Leonard (1985) in their article, and even indicate that the same cavity equation is used in both analyses. In both articles, the approach of Huang and Yu (1973) is cited as being used in the derivation of their equations. It is important to note that Huang and Yu (1973) work with constant-area ducts in their equations, and thus, one might conclude that Leonard (1985) and Lee and Liu (1989) have not interpreted the extension of this work to cavities with nonconstant areas identically. Wu et al. (1994) utilize the same cavity equation in their analysis of single-cavity dies with choker bars as in Lee and Liu (1989). Yuan (1995), in his analysis of dual-cavity dies, has recently proposed an inner cavity equation identical to that of Leonard (1995), except that he includes an additional term accounting for the momentum loss from the cavity into the slot.

To summarize, the slot equations utilized in the work cited previously (viscous-dominated as well as when inertial and gravitational effects are included) are typically based on one-dimensional flow there, and it is safe to say that such an approach has been widely accepted; there are a much smaller number of analyses that utilize two-dimensional Hele–Shaw approximations in the slot. Since the cavity equation is always one-dimensional when modeling is used, die performance is thus typically examined via a purely one-dimensional approach. Furthermore, when the flow in the cavity is viscous-dominated, the cavity equations cited earlier are identical; that is, with the exception of those analyses that incorporate axial viscous stresses, such as in Sartor (1990) and Durst et al. (1994). However, when such extra axial terms are removed, the same viscous-dominated equations do arise. On the other hand, when inertial effects are incorporated into the cavity equation, there are many proposed forms of the inertial term that arise. Additionally, the flow leakage of momentum into the slot (Yuan, 1995) is a further cavity equation modification that has been proposed.

To date, a formal approach to the derivation of the cavity and slot equations in the modeling approach has been lacking; that is, there has been no rigorous accounting of the order of magnitude of terms as well as the accompanying approximations to the full-blown three-dimensional equations. In this article, we derive the one-dimensional cavity and slot equations, utilizing asymptotic simplifications and integral techniques. In doing so, we rectify the differences between the proposed cavity equation forms. Furthermore, we place the one-dimensional slot-momentum and mass-conservation expressions used in previous studies on firm theoretical ground, specifically indicating the assumptions implicit in their use. We close this article with a discussion of the effect of the previously cited different forms of the cavity equation on predictions of pressure drop and flow in the cavity.

One-Dimensional Momentum Equation for Cavity

One-dimensional equation for a generalized Newtonian fluid

Our derivation begins with the complete set of three-dimensional equations that govern the flow in the cavity of the die (Figure 1). For the purposes of generality, we consider the momentum equation valid for generalized Newtonian fluids (Bird et al., 1977), of which the power-law fluid is a special case. For a generalized Newtonian fluid, the shear stress tensor, τ, is related to the rate of strain tensor, $\dot{\gamma}$, by the scalar viscosity, η, as

$$\tau = \eta\dot{\gamma}, \tag{1a}$$

where

$$\dot{\gamma} = \nabla V + (\nabla V)^T. \tag{1b}$$

In Eq. 1b, ∇ is the del operator, V is the velocity vector, and the superscript T denotes the transpose of the quantity in parentheses. The viscosity η in Eq. 1a depends on the magnitude of the rate of strain tensor for a generalized Newtonian fluid; for a Newtonian fluid, this viscosity is constant. The momentum equation, coupled with the usual continuity equation and boundary conditions, constitutes a well-posed system to solve for the cavity flow.

Simplifications to the governing equations can be obtained by considering the order-of-magnitude of terms. An enlarged cross-sectional view of the cavity geometry and coordinate system is shown in Figure 2, where the z-direction is oriented out of the figure, and corresponds to the z-direction in Figure 1. The cross-sectional area of the cavity is denoted as A, which is generally z-dependent, and the width of the die in the z-direction is denoted as W. Thus, we scale $z \sim W$, and choose $x, y \sim A_o^{1/2}$, where A_o is the cavity area at $z = 0$ (near the inlet in Figure 1a). As in the case of the cavity area, the volumetric flow rate in the cavity, Q, is z-dependent, since fluid must exit the cavity into the slot. We anticipate that the major component of flow in the cavity is in the z-direction, owing to the fact that the cavity is typically long compared to the cross-sectional area, A; let us scale $V_z \sim Q_o/A_o$, where Q_o is the volumetric flow rate delivered to the cavity at $z = 0$ (Figure 1a). Appropriate scales for the velocity components in the cross-sectional plane are chosen according to conservation of mass as $V_x, V_y \sim Q_o/(WA_o^{1/2})$. We scale the non-Newtonian viscosity as $\eta \sim \eta_0$, where η_0 is the Newtonian limiting viscosity at low rate of strain characteristic of generalized Newtonian fluids (Bird et al., 1977). With this viscosity scale, the pressure forces must balance the anticipated dominant flow in the cavity (in the z-direction); consequently, the pressure scales as $P \sim \eta_0 Q_o W/A_o^2$.

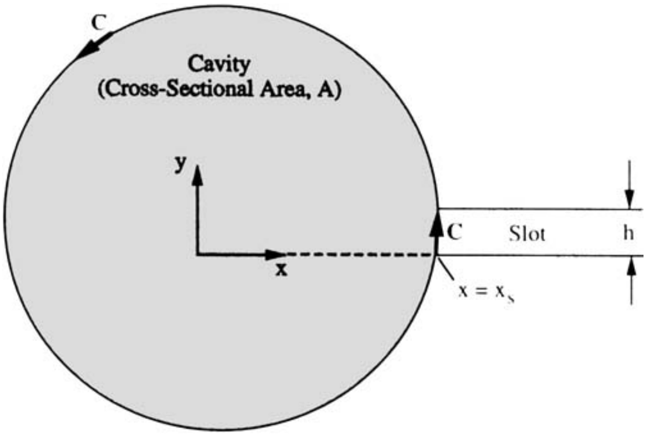

Figure 2. Detailed cross-sectional view of the cavity; the z-direction is oriented out of the figure.

With the preceding scalings, the following dimensionless parameters arise in the equations of motion:

$$\epsilon = \frac{A_o^{1/2}}{W}, \quad Re = \frac{\rho Q_o}{W\eta_0}, \quad G_x = \frac{\rho g_x A_o^2}{\eta_0 Q_o},$$
$$G_y = \frac{\rho g_y A_o^2}{\eta_0 Q_o}, \quad G_z = \frac{\rho g_x A_o^2}{\eta_0 Q_o}. \tag{2}$$

In Eq. 2, g_x, g_y, and g_z are the components of gravity in the x, y, and z directions. For typical die geometries, it is anticipated that the characteristic cross-sectional length scale of the cavity, $A_o^{1/2}$, is small compared to its width, W; consequently, the parameter ϵ will be small. Depending upon the particular physical situation, the other parameters in Eq. 2 may be large or small. For our purposes, we will consider them to be of order unity. We are thus led to consider the simplifications to the governing equations as $\epsilon \to 0$, holding the other parameters in Eq. 2 fixed. In this limit, we find that in the x and y-components of the equations of motion simplify greatly. The inertial terms are incorporated at $O(Re\epsilon^2)$, the viscous terms are of $O(\epsilon^2)$, and the gravitational terms are of $O(\epsilon G_x)$ and $O(\epsilon G_y)$. In the lowest-order approximation, then, the x and y components of the equation of motion indicate that there are no pressure variations across the cavity cross section (Figure 1b). The z-component of the equation of motion (i.e., in the principal direction of flow) indicates that inertial terms are of $O(Re)$ and gravity terms are $O(G_z)$. Additionally, we find that the viscous stresses associated with velocity gradients in the z-direction are $O(\epsilon^2)$.

With the simplifications just described, the lowest-order z-momentum equation is given in dimensional form as

$$\rho\left(V_x\frac{\partial V_z}{\partial x} + V_y\frac{\partial V_z}{\partial y} + V_z\frac{\partial V_z}{\partial z}\right) = -\frac{dP}{dz} + \frac{\partial}{\partial x}\left(\eta\frac{\partial V_z}{\partial x}\right) + \frac{\partial}{\partial y}\left(\eta\frac{\partial V_z}{\partial_y}\right) + \rho g_z, \tag{3a}$$

where

$$\eta = \eta(|\dot{\gamma}|), \qquad |\dot{\gamma}|^2 = \left(\frac{\partial V_z}{\partial x}\right)^2 + \left(\frac{\partial V_z}{\partial y}\right)^2. \tag{3b}$$

In Eq. 3b, note that the quantity $|\dot{\gamma}|$ is the magnitude of the approximated rate of strain tensor. The continuity equation is unaffected in form by the limiting process, and is given as

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 0. \tag{3c}$$

The appropriate kinematic and no-slip conditions to be applied are standard:

$$V_x = V_y = V_z = 0 \text{ on the solid boundaries.} \tag{3d}$$

The system Eq. 3 is the set of approximate equations that govern the flow of a generalized Newtonian fluid in the inner cavity. We acknowledge here that the system Eq. 3 is not well posed, in that all the velocity components explicitly appear, and additional higher-order momentum equations involving the x and y velocity components, analogous to Eq. 3a, are required. In fact, the rigorous solution for the velocity components is coupled. However, it is our intent to obtain the one-dimensional set of governing equations for the cavity, and for this procedure, the system Eq. 3 is entirely adequate.

Continuing with our derivation, we now average the simplified governing equations across the cavity cross section, to obtain an equivalent integral expression. In doing so, recall that the cross-sectional area of the cavity, A, is in general z-dependent (Figure 1). Integrating Eq. 3a over A at a given z, that is, the shaded region shown in Figure 2, the result is

$$\rho \int_A \left(V_x \frac{\partial V_z}{\partial x} + V_y \frac{\partial V_z}{\partial y} + V_z \frac{\partial V_z}{\partial z} \right) dS = -\frac{dP}{dz} A + \rho g_z A + \int_A \left(\frac{\partial}{\partial x} \left(\eta \frac{\partial V_z}{\partial x} \right) + \frac{\partial}{\partial y} \left(\eta \frac{\partial V_z}{\partial y} \right) \right) dS, \tag{4}$$

where dS is the differential area element, and η is given by Eq. 3b. Before proceeding further, we must acknowledge that the scalings used to obtain the system Eq. 3 are valid everywhere in the cavity, *except* in the vicinity of the entrance to the slot (i.e., near $x = x_s$ in Figure 2). This is because the slot height, h, is typically extremely small with respect to the characteristic cross-sectional dimension of the cavity, $A^{1/2}$. Thus, in the vicinity of the slot, the velocities in the x- and y-directions must become important so as to accommodate the loss in fluid flow from the cavity into the slot. In principle, then, we cannot integrate over the total cross-sectional area of the inner cavity. However, since $h \ll A^{1/2}$, the area over which the system Eq. 3 is not valid is small compared with the cross-sectional area of the cavity; thus, anticipating that velocities are bounded near the slot, we can, with little incurred error, integrate over the whole cavity area, up to the slot.

We now simplify the lefthand side of Eq. 4. Employing integration by parts, we find:

$$\int_A V_x \frac{\partial V_z}{\partial x} dS = \int_C V_z V_x dy - \int_A V_z \frac{\partial V_x}{\partial x} dS \tag{5a}$$

$$\int_A V_y \frac{\partial V_z}{\partial y} dS = \int_C V_z V_y dx - \int_A V_z \frac{\partial V_y}{\partial y} dS, \tag{5b}$$

where the first integral on the righthand side of Eqs. 5a and 5b is a line integral over the contour C shown in Figure 2. Now, along the solid boundaries, all velocities are zero. Thus, the only contribution to the line integrals in Eq. 5 comes in the vicinity of the slot, that is, at $x = x_s$ in Figure 2:

$$\int_C V_z V_x dy = \int_0^h V_x V_z \big|_{x=x_s} dy. \tag{5c}$$

In addition, since $dx = 0$ along the contour associated with $x = x_s$ in Figure 2,

$$\int_C V_z V_y dx = 0. \tag{5d}$$

Substituting Eq. 5 into the integral on the lefthand side of Eq. 4, we obtain

$$\int_A \left(V_x \frac{\partial V_z}{\partial x} + V_y \frac{\partial V_z}{\partial y} + V_z \frac{\partial V_z}{\partial z} \right) dS = \int_0^h V_x V_z \big|_{x=x_s} dy + \int_A V_z \left(\frac{\partial V_z}{\partial z} - \frac{\partial V_x}{\partial x} - \frac{\partial V_y}{\partial y} \right) dS. \tag{6}$$

Now, employing the continuity equation, Eq. 3c, in Eq. 6 and substituting the result into Eq. 4, yields

$$\rho \left(\int_A \frac{\partial V_z^2}{\partial z} dS + \int_0^h V_x V_z \big|_{x=x_s} dy \right) = -\frac{dP}{dz} A + \rho g_z A + \int_A \left(\frac{\partial}{\partial x} \left(\eta \frac{\partial V_z}{\partial x} \right) + \frac{\partial}{\partial y} \left(\eta \frac{\partial V_z}{\partial y} \right) \right) dS. \tag{7}$$

To obtain a one-dimensional equation, Eq. 7 is solved approximately. To do so, we assume a trial function that satisfies:

$$\frac{\partial}{\partial x} \left(\eta \frac{\partial V_z}{\partial x} \right) + \frac{\partial}{\partial y} \left(\eta \frac{\partial V_z}{\partial y} \right) = -D(z), \tag{8a}$$

where $D(z)$ is an as of yet undetermined function. Clearly, a boundary condition appropriate for this trial function is $V_z = 0$ on the solid walls of the cavity (from condition 3d), that is, on the wall portions of the contour C in Figure 2. However, to fully specify the form of our trial function, we must also specify an appropriate boundary condition along the portion of C at $x = x_s$ (at the slot) in Figure 2. Such a condition is obtained by consideration of the magnitude of the second integral on the lefthand side of Eq. 7. This estimate, given in Appendix A, indicates that the integral is small compared to the other terms in Eq. 7; consequently, it can be neglected with little incurred error. We are thus justified in choosing a trial function that satisfies $V_z = 0$ at $x = x_s$, which is consistent with the neglect of this line integral. Thus, the contour C in Figure 2 can be treated as if it consisted only of a solid wall, and

$$V_z = 0 \quad \text{on} \quad C. \tag{8b}$$

Thus, for example, if the cavity area has a circular cross section connected to a slot having a small slot height, we treat the cavity, for the purposes of solving Eq. 8a, as if the slot were not there, and the cavity wall had no slot.

With the trial function defined by Eqs. 8a and 8b, the integral equation 7 then becomes

$$\rho \frac{d}{dz} \int_A V_z^2 \, dS = -\left(\frac{dP}{dz} - \rho g_z + D(z) \right) A. \qquad (8c)$$

We note here that the cross-sectional area A, can, in general, vary with z. Thus, in pulling the z-derivative out of the area integral on the lefthand side of Eq. 7, Leibnitz's rule has been used, which also utilizes the boundary condition, Eq. 8b. Noting that the volumetric flow in the cavity is given by

$$Q(z) = \int_A V_z \, dy, \qquad (8d)$$

the functions $D(z)$ and V_z in Eq. 8 can be related to $Q(z)$, and once this relationship is established, Eq. 8c yields an equation relating the widthwise pressure gradient in the cavity to the volumetric flow rate. The result, Eq. 8, is the most general form of the one-dimensional equation governing the flow in the cavity for a generalized Newtonian fluid.

1-D cavity equation for a power-law fluid

We now apply Eq. 8 to obtain the one-dimensional equation for a power-law fluid, whose viscosity depends on the rate of strain as

$$\eta = m |\dot{\gamma}|^{n-1}, \qquad (9)$$

where m (the consistency coefficient) and n (the power-law index) are constants, and $|\dot{\gamma}|$ is magnitude of the rate-of-strain tensor given in Eq. 1b. To proceed, the trial function equation, Eq. 8a, is made dimensionless with z-varying scales Q and A. Natural scales are $V_z \sim Q/A$, where Q is the volumetric flow rate given by Eq. 8d, and $x,y, \sim A^{1/2}$. Based on the form of the viscosity and rate of strain given, respectively, by Eqs. 9 and 3b, we choose $\eta \sim m(Q^2/A^3)^{(n-1)/2}$. With these scales, Eq. 8a becomes

$$\frac{\partial}{\partial \bar{x}} \left(\bar{\eta} \frac{\partial \bar{V}_z}{\partial \bar{x}} \right) + \frac{\partial}{\partial \bar{y}} \left(\bar{\eta} \frac{\partial \bar{V}_z}{\partial \bar{y}} \right) = -\frac{D(z) A^{(1+3n)/2}}{mQ^n}, \qquad (10a)$$

while the viscosity becomes, from Eqs. 9 and 3b:

$$\bar{\eta} = \left[\left(\frac{\partial \bar{V}_z}{\partial \bar{x}} \right)^2 + \left(\frac{\partial \bar{V}_z}{\partial \bar{y}} \right)^2 \right]^{(n-1)/2}. \qquad (10b)$$

In Eq. 10, dimensionless variables are denoted with an overbar. A solution to Eq. 10 can be found by assuming:

$$\bar{V}_z = \bar{g}(z) \bar{f}(\bar{x}, \bar{y}) \qquad (11a)$$

where $\bar{f}$ and $\bar{g}$ are dimensionless functions. Substituting Eq. 11a into Eq. 10, the result is

$$\bar{g}(z) = \left(\frac{D(z) A^{(1+3n)/2}}{mQ^n} \right)^{1/n}, \qquad (11b)$$

where $\bar{f}$ satisfies

$$\frac{\partial}{\partial \bar{x}} \left(\bar{\eta}' \frac{\partial \bar{f}}{\partial \bar{x}} \right) + \frac{\partial}{\partial \bar{y}} \left(\bar{\eta}' \frac{\partial \bar{f}}{\partial \bar{y}} \right) = -1 \qquad (12a)$$

$$\bar{\eta}' = \left[\left(\frac{\partial \bar{f}}{\partial \bar{x}} \right)^2 + \left(\frac{\partial \bar{f}}{\partial \bar{y}} \right)^2 \right]^{(n-1)/2}, \qquad (12b)$$

and, from Eq. 8b,

$$\bar{f} = 0 \quad \text{on} \quad \bar{C}. \qquad (12c)$$

Note that from the chosen scalings, $\bar{A} = 1$, regardless of the shape of the domain.

To relate the function $D(z)$ to Q, we utilize the volumetric flow relation given by Eq. 8d; scaling it and employing Eq. 11, the result is

$$D(z) = \frac{mQ^n}{K^n A^{(1+3n)/2}}, \qquad (13a)$$

where

$$K = \int_{\bar{A}} \bar{f}(\bar{x}, \bar{y}) \, d\bar{x} d\bar{y}. \qquad (13b)$$

In Eq. 13b, K is in general a function of z, and is the *viscous shape factor* referred to in the Introduction (Miller, 1972; Liu, 1983); it can be viewed as the dimensionless volumetric flow rate through the domain. The viscous shape factor depends clearly on the *shape* of the domain, since the solution for $\bar{f}$ must satisfy Eq. 12, and the dimensionless area, $\bar{A}$, is always equal to 1; thus, the z-dependence in Eq. 13b is invoked because for a general cavity geometry, the shape of the domain may vary in the z-direction. Additionally, the viscous shape factor is also a function of the power-law index, n, as is evidenced by the system Eq. 12. However, if we restrict attention to cavities that have cross-sectional shapes that do not vary with z, then K in Eq. 13b is a constant for a given fluid rheology (i.e., value of n). Thus, for a given domain shape and rheology, Eq. 12 need only be solved once to obtain $\bar{f}(\bar{x}, \bar{y})$, and from Eq. 13b, a constant value for K is obtained (Liu, 1983). Note that the preceding simplification does not restrict the magnitude of the cavity *area* to be constant, since the cavity area has been scaled out of the definition for K.

With the shape factor defined, and using the definition Eq. 13a, the dimensionless velocity, Eq. 11, is written as

$$\bar{V}_z = \frac{1}{K} \bar{f}(\bar{x}, \bar{y}). \qquad (14)$$

With the assumption of a constant cross-sectional shape, then, Eq. 14 indicates that $\bar{V}_z$ is not a function of z. Expressing the lefthand side of Eq. 8c in terms of $\bar{V}_z$, the result is

$$\rho \frac{d}{dz}\int_A V_z^2 dS = \rho \frac{d}{dz}\int_{\bar{A}} \bar{V}_z^2 \frac{Q^2}{A} d\bar{S} = \rho\beta \frac{d}{dz}\left(\frac{Q^2}{A}\right), \quad (15a)$$

where β is the *kinetic shape factor* for the power-law fluid given by

$$\beta = \int_{\bar{A}} \bar{V}_z^2 d\bar{S} = \frac{1}{K^2}\int_{\bar{A}} \hat{f}(\bar{x},\bar{y})^2 d\bar{x}\, d\bar{y}. \quad (15b)$$

We note here that the kinetic shape factor given by Eq. 15b is precisely that referred to in the introduction (as given by Leonard, 1985; Lee and Liu, 1989). Substituting Eqs. 15 and 13a into Eq. 8c, the result is

$$\rho\beta \frac{1}{A}\frac{d}{dz}\left(\frac{Q^2}{A}\right) + \frac{mQ^n}{K^n A^{(1+3n)/2}} - \rho g_z = -\frac{dP}{dz}. \quad (16)$$

Equation 16 is the desired one-dimensional equation, relating the volumetric flow rate to the pressure gradient in a cavity, for a power-law fluid; shape factors in this equation are functions of cavity shape and power-law index, according to Eqs. 13b and 15b. From the viscosity dependence in Eq. 9, it is clear that the Newtonian one-dimensional equation can be obtained from Eq. 16 by setting $n = 1$ and interpreting m as the constant Newtonian viscosity, which we denote as η_0 (i.e., $m = \eta_0$ when $n = 1$). For a Newtonian fluid, then, the shape factor parameters are only a function of the shape of the domain.

Comments on constant shape factors

In the preceding derivation, the one-dimensional cavity equation is greatly simplified because of the ability to define a *single* kinetic and viscous shape factor; that is, for a cavity of a given cross-sectional shape and a given power-law index. With the importance of this fact, some comments are in order. The kinetic and viscous shape factors are obtained by utilizing the solution to the system Eq. 12; this solution corresponds to velocity profiles describing fully developed pressure-driven viscous flow in a duct of constant cross section. Thus, the viscous and kinetic shape factors themselves are derived under an implicit assumption that the velocity field in the cavity has a fully developed form. Based on the asymptotic analysis performed for the cavity flow, the fully developed assumption will not be valid in the vicinity of the cavity ends. This is apparent by inspection of Eq. 3a, in that the viscous term involving the z-derivative does not appear (it does appear in the original equation before approximation); consequently, the ability to satisfy fully three-dimensional boundary conditions at the ends of the domain in the z-direction is lost. Clearly, the neglect of end effects is necessary for the assumed fully developed profile to even be qualitatively correct. We also note that the fully developed assumption will be less good when inertial effects become large in the cavity, since a velocity field assuming a balance of viscous and pressure forces is implicitly used in the determination of the shape factors.

It is well known that the power-law model has an important shortcoming, in that it cannot limit to a constant Newtonian viscosity at low rates of strain. There are generalized Newtonian rheological models that do not have this shortcoming, such as the Carreau model (Bird et al., 1977). However, even the small increase in mathematical complexity of these forms, compared with the power-law model, makes the one-dimensional approach prohibitive. To see why, we briefly consider the application of Eq. 8 for a Carreau fluid. The relation of viscosity, η, to the rate of strain, $|\dot{\gamma}|$, for the Carreau model is given as

$$\eta - \eta_\infty = (\eta_0 - \eta_\infty)[1 + (\lambda|\dot{\gamma}|)^2]^{(n-1)/2}, \quad (17)$$

where η_0 and η_∞ are the viscosities at small and large rates of strain, respectively, n is the power-law index, and λ is the relaxation time.

Following the previous approach for a power-law fluid, we make variables in Eq. 8a dimensionless using the same scalings, except from the form Eq. 17, we scale $\eta \sim \eta_0$. With these scales, Eq. 8a becomes

$$\frac{\partial}{\partial \bar{x}}\left(\bar{\eta}\frac{\partial \bar{V}_z}{\partial \bar{x}}\right) + \frac{\partial}{\partial \bar{y}}\left(\bar{\eta}\frac{\partial \bar{V}_z}{\partial \bar{y}}\right) = -\frac{D(z)A^2}{\eta_0 Q}, \quad (18a)$$

while the viscosity Eq. 17 is

$$\bar{\eta} = \bar{\eta}_\infty + (1 - \bar{\eta}_\infty)\left[1 + (\bar{\lambda}|\bar{\dot{\gamma}}|)^2\right]^{(n-1)/2}, \quad (18b)$$

where

$$\bar{\eta}_\infty = \frac{\eta_\infty}{\eta_0}, \quad \bar{\lambda} = \frac{\lambda Q}{A^{3/2}} \quad (18c)$$

and

$$|\bar{\dot{\gamma}}|^2 = \left(\frac{\partial \bar{V}_z}{\partial \bar{x}}\right)^2 + \left(\frac{\partial \bar{V}_z}{\partial \bar{y}}\right)^2. \quad (18d)$$

Now, following the approach for a power-law fluid, we look for a solution to Eq. 18 of the form given in Eq. 11a, which separates the z-dependence from the x- and y-dependence. However, upon substitution, we find that a solution cannot be found in this form. The ramifications of the inability to separate out the z-dependence are profound. This can be seen by recalling the implications of this form to the derivation of the one-dimensional equation for a power-law fluid. By using the form in Eq. 11, the system Eq. 12 was obtained, which allowed for the x- and y-dependent part of the velocity $\bar{V}_z$ to be determined, independent of the z-dependent part. Further, this allowed for the identification of a shape factor, K, in Eq. 13, which was a constant for a given domain shape and power-law index. For the Carreau case considered earlier, the inability to use the form Eq. 11a means that system Eq. 18 must be solved *at each value of z*. Consequently, through the volumetric flow relation, Eq. 8c, there is a different relationship between $D(z)$ and Q at each z location; that is, a simple relation, such as that given in Eq. 13a, that relates these two quantities is not available. Effectively, then, there is no single viscous or kinetic shape factor that can be used to characterize a given cavity shape for a given rheology, or equivalently, the shape factor varies with z along the cavity, even for a given cavity shape and rheology. This greatly complicates the use of the one-dimensional cavity equation, compared with the power-law case. For this reason, Yuan (1995) has proposed a truncated power-law model, which connects Newtonian and power-law behavior (Bird et al., 1977), to overcome the shortcoming of the straight power-law model, but yet retain the simplicity of the one-dimensional cavity equation with constant shape factors.

Finally, we note here that Liu and Hong (1988) have identified viscous shape factors for rheological models which have forms similar to the Carreau model, namely, the Ellis and Herschel–Buckley models. Such models also do not allow for the separability of the z- and x-y-dependences in the trial function as well. Based on the preceding arguments, the shape factors of Liu and Hong (1988) for these models can only rigorously be used for duct flows in which $\dot{Q}$ and A are absolute constants; such a restriction limits their usefulness in the one-dimensional analysis of die flows.

1-D Momentum Equation for the Slot

Derivation for a generalized Newtonian fluid

As in the case of the cavity equation, our derivation begins with the complete set of three-dimensional equations that govern the flow of a generalized Newtonian fluid (as defined by Eq. 1) in the slot region of Figure 1. Simplifications to these equations can be obtained by considering the order of magnitude of terms. As is shown in the figure, the slot itself has the same width as the cavities, that is, it is of width W in the z-direction and has a constant slot height h. Additionally, as shown in Figure 1, the slot length L can in general vary with z; in constructing appropriate scales, we use the slot length at $z=0$, denoted as L_o. Figure 3 gives a more detailed cross-sectional view of the slot region to be analyzed (the shaded region), where the z-direction is oriented out of the figure, and corresponds to the z-direction in Figure 1. As shown in Figure 3, the beginning of the slot is located at $x = x_s$. Thus, we will scale $x - x_s \sim L_o$, while $y \sim h$ and $z \sim W$. All the fluid that is introduced into the inner cavity at $z=0$, denoted as Q_o, must flow through any cross-sectional area of the slot in the y-z plane (Figure 1). Thus, overall mass conservation indicates that $V_x \sim Q_o/(Wh)$. Furthermore, scales for the other velocity components are obtained by balancing all terms in the continuity equation, which yields $V_y \sim Q_o/L_o$ and $V_z \sim Q_o W/(hL_o)$. We scale the generalized Newtonian viscosity as $\eta \sim \eta_0$, where η_0 is the Newtonian viscosity at low rates of strain. Since it is the pressure drop across the slot that drives the flow away from the cavity, we scale the slot pressure, P', as $P' \sim \eta_0 Q_o L_o/(Wh^3)$ (the prime is utilized to distinguish the slot pressure from the unprimed cavity pressure for later reference).

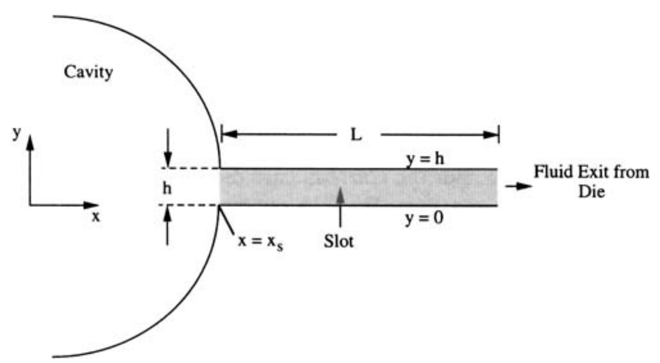

Figure 3. Detailed cross-sectional view of the slot. The z-direction is oriented out of the figure.

With the preceding scalings, the following dimensionless parameters arise in the equations of motion:

$$\alpha = \frac{h}{L_o}, \quad \delta = \frac{L_o}{W}, \quad Re = \frac{\rho Q_o}{W\eta_0}, \quad G'_x = \frac{\rho g_x W L_o^3}{\eta_0 Q_o},$$

$$G'_y = \frac{\rho g_y W L_o^3}{\eta_0 Q_o}, \quad G'_z = \frac{\rho g_z W L_o^3}{\eta_0 Q_o}, \tag{19}$$

where g_x, g_y, and g_z are the components of gravity in the x, y, and z directions. For typical die geometries, it is anticipated that the characteristic height of the slot, h, is small with respect to the slot length scale L_o; consequently, the parameter α is small. Additionally, if we restrict attention to dies that are wide compared with the slot length, then the parameter δ in Eq. 19 is small as well. Depending upon the particular physical situation, the other parameters in Eq. 19 may be large or small. For our purposes, we will consider them to be of order unity. We are thus led to consider the simplifications to the governing equations as $\alpha \to 0$ and $\delta \to 0$, holding the other parameters in Eq. 19 fixed. In this limit, we find that the momentum equation simplifies greatly. In the x and z components, inertial terms are $O(Re\,\alpha)$, velocity gradients in the x and z directions are of $O(\alpha^2)$ or smaller, and the gravitational terms are of $O(\alpha^3 G'_x)$ and $O(\alpha^3 \delta G'_z)$. All other terms in the x-component equation are of order unity. In the lowest-order approximation, then, pressure forces balance the viscous stresses due to the variations in V_x across the slot gap. An additional simplification is obtained in the z-component equation, in that the pressure gradient term is of $O(\delta^2)$. This means that in the lowest order, the only term arising in the z-component equation is the viscous gradient across the gap. Finally, in the y-component of the momentum equation, we find that inertial effects are of $O(Re\,\alpha^3)$, all viscous effects are of $O(\alpha^2)$ or smaller, and gravitational effects are $O(\alpha^4 G'_y)$; consequently, there are no pressure gradients in the y-direction (across the slot gap) to lowest order.

With the simplifications just described, the lowest order components of the momentum equation are given in dimensional form as

$$\frac{\partial}{\partial y}\left(\eta \frac{\partial V_x}{\partial y}\right) = \frac{\partial P'}{\partial x} \tag{20a}$$

$$\frac{\partial P'}{\partial y} = 0 \tag{20b}$$

$$\frac{\partial}{\partial y}\left(\eta \frac{\partial V_z}{\partial y}\right) = 0 \tag{20c}$$

$$\eta = \eta(|\dot{\gamma}|), \qquad |\dot{\gamma}|^2 = \left(\frac{\partial V_x}{\partial y}\right)^2 + \left(\frac{\partial V_z}{\partial y}\right)^2. \tag{20d}$$

The continuity equation is unaffected in form by the limiting process, and is given as

$$\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}=0. \tag{20e}$$

The appropriate kinematic and no-slip conditions to be applied are standard:

$$V_x=V_y=V_z=0 \quad \text{at} \quad y=0,1. \tag{20f}$$

Additional boundary conditions, necessary to completely determine the pressure and velocity fields in the slot, are obtained by linking the cavity and the slot regions; such conditions are presented following our analysis of the slot flow.

The solution of the system Eq. 20 yields:

$$V_y=V_z=0 \qquad \text{for all } x,y,z \text{ in the slot.} \tag{21}$$

Additionally, we find that the most general form of the pressure field in the slot can be expressed as

$$P'=F(z)(x-x_s)+G(z), \tag{22}$$

where $F(z)$ and $G(z)$ are as of yet undetermined functions. The velocity field in the slot is obtained as the solution of Eq. 20a; this equation can be simplified by a single integration to yield

$$\left(\eta\frac{\partial V_x}{\partial y}\right)=F(z)\left(y-\frac{h}{2}\right), \tag{23a}$$

where

$$\eta=\eta(|\dot{\gamma}|), \qquad |\dot{\gamma}|^2=\left(\frac{\partial V_x}{\partial y}\right)^2. \tag{23b}$$

In writing Eq. 23a, we have utilized the fact that Eq. 20f implies that the velocity field is symmetric about $y=h/2$. The remaining boundary condition to be applied is

$$V_x=0 \quad \text{at} \quad y=0. \tag{23c}$$

We note here that Eqs. 22–23 are the standard equations governing unidirectional flow in a slot; that is, *except* for the z-dependence, which is explicitly indicated.

The form of the function $F(z)$ in Eqs. 22 and 23a is determined by consideration of the volumetric flow rate per width in the slot, $q(z)$, which defined as

$$q(z)=\int_0^h V_x\,dy. \tag{23d}$$

For perfect widthwise uniformity in the slot, note that $q=Q_o/W$ for all z. Clearly, from Eqs. 23a and 23d the function $F(z)$ is related to $q(z)$, its precise form depending on the generalized Newtonian rheological model. At this point, the function $G(z)$ in Eq. 22 has not been determined, as it is obtained through the linkage of the cavity and slot flows to be considered later in this article. The pressure field, Eq. 22, is the desired one-dimensional equation in the slot for a generalized Newtonian fluid.

At this point, we comment that the preceding results are contingent on the fact that the parameter δ in Eq. 19 is small. In a different limiting process where $\alpha\to 0$ and where $\delta=O(1)$ (i.e., $W\sim L_o$) while holding all other parameters fixed in Eq. 19, the neglected pressure gradient term $\partial P'/\partial z$ will now appear in Eq. 20c. After simplification, the system Eq. 20 leads directly to the well-known Hele–Shaw equations (Schlichting, 1979) if a Newtonian fluid is assumed (i.e., $\eta=$ constant). In such a case, the pressure field in the slot satisfies LaPlace's equation, that is,

$$\frac{\partial^2 P'}{\partial x^2}+\frac{\partial^2 P'}{\partial z^2}=0. \tag{24a}$$

It is apparent by inspection of Eq. 24a that when the scales for x and z are used ($x\sim L_o$, $z\sim W$), the following order-of-magnitude estimate holds:

$$\frac{\dfrac{\partial^2 P'}{\partial z^2}}{\dfrac{\partial^2 P'}{\partial x^2}}=\frac{L_o^2}{W^2}=\delta^2. \tag{24b}$$

Thus, in the limit as $\delta\ll 1$, the first term in Eq. 24a becomes dominant, which implies that the pressure field is linear in x, as given precisely by Eq. 22; this is a further demonstration of the validity of Eqs. 22 and 23, since these equations must be valid for the special case of a Newtonian fluid.

1-D slot equation for a power-law fluid

For many generalized models, such as the Carreau fluid governed by Eq. 17, a numerical solution is required for the system Eq. 23. However, for a power-law fluid, the system Eq. 23 can be solved analytically. For a power-law fluid, the dependence of viscosity on strain in the slot is given by

$$\eta=m\left|\frac{\partial V_x}{\partial y}\right|^{n-1}, \tag{25}$$

where the vertical lines denote absolute value. Substituting Eq. 25 into the system Eq. 23 and solving yields

$$F(z)=-\frac{m2^{n+1}}{h^{1+2n}}\left(\frac{1}{n}+2\right)^n q(z)^n, \tag{26a}$$

and thus

$$V_x=\frac{q(z)}{h}\left(\frac{1+2n}{1+n}\right)\left[1-\left|\frac{2y}{h}-1\right|^{(1+n)/n}\right]. \tag{26b}$$

The pressure field in the slot is obtained by substituting Eq. 26a into Eq. 22 to yield

$$P' = -\left[\frac{m2^{n+1}}{h^{1+2n}}\left(\frac{1}{n}+2\right)^n q(z)^n\right](x-x_s)+G(z). \quad (27)$$

The result, Eq. 27, is the one-dimensional slot equation for a power-law fluid. The function $G(z)$ is not known at this point and is obtained by linking the cavity and slot, which is performed in the next section.

Linking the 1-D Cavity and Slot Equations

Conservation of mass

We begin by noting that both the slot and cavity one-dimensional equations derived previously are not valid in the immediate vicinity of the entrance to the slot, based on the asymptotic analyses utilized. As shown in Figure 4, there is an intermediate region that links the slot and cavity regions; we will link the cavity and slot equations, through this intermediate region, by an integral approach. The integral approach is consistent with the one-dimensional approximations utilized thus far.

The continuity equation, Eq. 3c, is valid in the cavity, intermediate, and slot regions shown in Figure 4. Let us integrate the continuity equation over the shaded cross-sectional region shown in Figure 4, which extends from the cavity, through the intermediate region, and ends at the beginning of the slot region (the slot region begins where Eq. 26b is satisfied, that is, at the righthand boundary of the intermediate region shown in Figure 4). Denoting A' as the cross-sectional area of the cavity (excluding the small intermediate-region area that extends into the cavity as shown in Figure 4) and A_I as the area of the intermediate region, the integrated continuity equation can be written in dimensional form as

$$\int_{A'+A_I}\frac{\partial V_z}{\partial z}dS = \int_{A'+A_I} -\left(\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}\right)dS. \quad (28a)$$

Using Green's theorem, the righthand side of Eq. 28a can be expressed in terms of a line integral as

$$\int_{A'+A_I}\left(\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}y\right)dS = \int_C (V_x dy + V_y dx), \quad (28b)$$

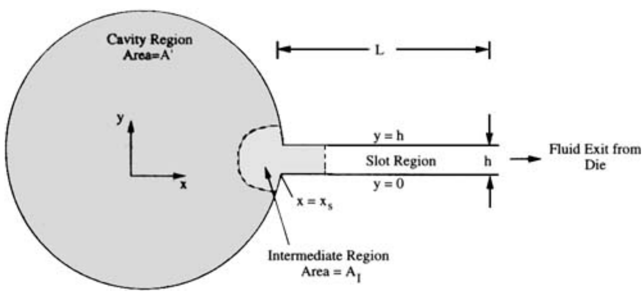

Figure 4. Cross-sectional view of the intermediate region, which joins the cavity and slot regions of the die; the z-direction is oriented out of the figure.

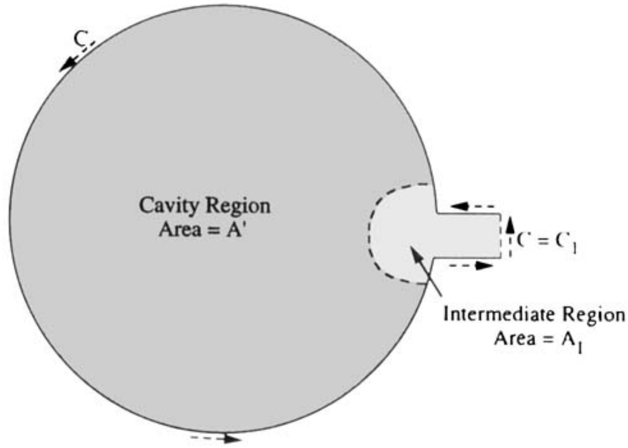

Figure 5. Contour C used in the integration of the continuity equation.

where C is oriented counterclockwise as shown in Figure 5 (which corresponds to the shaded domain shown in Figure 4). Now, along all the solid boundaries, $V_x = V_y = 0$, except along the portion of the contour C labeled as C_1, that is, along the righthand boundary of the intermediate region in Figure 5. Consequently, using the local volumetric flow per width in the slot, defined by Eq. 23d, Eq. 28b becomes

$$\int_{A'+A_I}\left(\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}\right)dS = \int_0^h V_x dy = q(z). \quad (28c)$$

We now simplify the lefthand side of Eq. 28a. We would like to pull the z-derivative outside of the integral in Eq. 28a, so the resulting integrand can be related to the volumetric flow rate in the domain, but we acknowledge that the cavity area may vary with z. Thus, we must employ Leibnitz rule, which requires that the value of the velocity V_z be known along all boundaries of the domain $A' + A_I$. Certainly, $V_z = 0$ along the solid boundaries. We have also established that $V_z \ll 1$ along the contour C_1 (Figure 5) according to Eq. 21, with respect to the velocity scaling used in the slot. However, the velocity scalings in the slot and cavity are different, and so we need to relate the magnitudes of the velocity in the slot and cavity to choose the correct approximate velocity to use along the contour C_1. Such a relationship is estimated in Appendix B, where it is found that

$$V_z|_{\text{slot}} \sim \frac{h^2}{A} V_z|_{\text{cavity}}. \quad (29)$$

Clearly, since $h^2 \ll A$ for typical dies, the relation 29 indicates that the velocity in the slot region along contour C_1 in Figure 5 is negligible with respect to the flow in the cavity; thus Leibnitz's rule implies that:

$$\int_{A'+A_I}\frac{\partial V_z}{\partial z}dS \sim \frac{\partial}{\partial z}\int_{A'+A_I} V_z dS. \quad (30a)$$

Now, the cross-sectional area of the intermediate region (see Figure 4), A_I, is smaller than the cross-sectional area of the slot itself. Based on results of Atkinson et al. (1969), the entry length into the slot, which influences the size of the intermediate region, is no more than a few slot heights (h) at moderate Reynolds numbers. It is thus reasonable to bound the area of this region as $A_I \ll hL$. Based on the previous limits and typical die geometries, it is also anticipated that

$$\frac{A_I}{A'} \sim \frac{hL}{A'} \ll 1, \tag{30b}$$

and since V_z in the intermediate region is approximately bounded by the scales in the cavity and slot, we can neglect the contribution over A_I in the integral 30a. Additionally, we can replace A' with A, the total area of the inner cavity up to the slot (this area is shown by the shaded region in Figure 2), without incurring further error, since A_I is neglected. Thus,

$$\frac{\partial}{\partial z} \int_{A' + A_I} V_z \, dS \sim \frac{\partial}{\partial z} \int_A V_z \, dS. \tag{30c}$$

From Eq. 8d, we immediately identify the integral on the righthand side of Eq. 30c with the local volumetric flow rate in the cavity, $Q(z)$; using this fact, and substituting Eqs. 30c and 28c into Eq. 28a, the result is

$$\frac{dQ(z)}{dz} = -q(z). \tag{31}$$

Thus, subject to the preceding assumptions, the leakage of flow into the inner slot is accompanied by a local change in the volumetric flow in the inner cavity and is the statement of mass conservation for the cavity–slot system.

Linking the cavity and slot pressures

We now relate the pressure in the cavity and the pressure in the slot. In the cavity, we established previously that the pressure is only a function of z to lowest order, that is,

$$P = P(z). \tag{32}$$

From Eq. 22, the pressure in the slot is always a linear function of x, where as $x \to x_s$, the slot entrance is approached. Now, to link Eqs. 32 and 22, we anticipate that the length of the intermediate region (which separates the regions of validity of these two equations) is only a few slot heights long at moderate Reynolds number (Atkinson et al., 1969) and is small compared with L, the length of the slot. Furthermore, the pressure scale in the intermediate region is approximately the same as that in the slot. Based on this reasoning, we anticipate that the pressure drop across the intermediate region is small compared with the total pressure drop across the slot, and so we can equate the cavity and slot pressures at $x = x_s$; thus, $G(z) = P(z)$ in Eq. 22, and assuming that the pressure at the exit of the die slot (located at $x = x_s + L$) is zero, that is, $P' = 0$, we obtain

$$P(z) = F(z)L, \tag{33}$$

where $F(z)$ is obtained for a generalized fluid from Eq. 23. The result, Eq. 33, gives an expression for the cavity pressure in terms of the slot flow. For a power-law fluid, Eq. 33 can be written, using Eq. 26a, as

$$P(z) = \left[\frac{m2^{n+1}}{h^{1+2n}} \left(\frac{1}{n} + 2\right)^n q(z)^n\right] L. \tag{34}$$

We note here that in Eqs. 33 and 34, L is in general a function of z, as indicated in Figure 1.

Before leaving our consideration of the linkage of the cavity and slot pressures, we note that the pressure scalings used to derive the cavity and slot equations are different. We must make sure that the order of magnitude of the cavity and slot pressures are consistent to establish the self-consistency of the derived results Eqs. 33 and 34. The pressure scale in the slot, P_s, is given $P_s \sim \eta_0 Q_o L_o/(Wh^3)$, while the pressure scale used in our cavity analysis, P_c, is given as $P_c \sim \eta_0 Q_o W/A_o^2$. The relationship between these scales is written as:

$$P_c \sim \frac{W^2 h^3}{L_o A_o^2} P_s. \tag{35a}$$

For anticipated die geometry requirements, Eq. 35a suggests $P_c \ll P_s$, which would seem to indicate that the slot and cavity pressures are of a different order of magnitude and cannot interact as indicated in Eqs. 33 and 34. The resolution of this problem is obtained by noting that the pressure scale of the inner cavity was obtained by balancing the pressure gradient against the anticipated dominant viscous flow, which is in the die-width direction. In addition, we note that the absolute pressure level in the cavity itself does nothing to drive flow, since pressure differences drive flow (which was how the inner cavity pressure scale was chosen in the first place). Let us assume that the cavity pressure in Eq. 33 can be written as

$$P(z) = P_R + J(z), \tag{35b}$$

where P_R is the average pressure in the cavity, and $J(z)$ is some arbitrary function contributing to flow nonuniformity. Then according to the scalings leading to Eq. 35a:

$$P_R \gg J(z). \tag{35c}$$

However, it is clear that

$$\frac{dP(z)}{dz} = \frac{dJ(z)}{dz}. \tag{35d}$$

Thus, the pressure level in the cavity is determined by the slot, but the pressure gradient in the cavity balances the flow occurring there. In this way, the disparate pressure scales can be rectified. Our linkage of the cavity and slot pressures is contingent on the validity of Eq. 35. With this final point considered, Eq. 34 is the appropriate one-dimensional equation for the slot for a power-law fluid.

Discussion and Conclusions

We now make some specific comments regarding the derived cavity, slot, and mass-conservation equations. The mass-conservation equation, Eq. 31, has been utilized in virtually all single-cavity flow analyses. The one-dimensional slot equation for a power-law fluid, Eq. 34, has been similarly utilized, with the exception of those analyses in which a Hele–Shaw slot flow was assumed; examples of such literature, cited in the Introduction, are Sartor (1990), Vrahopoulou (1991, 1992). As discussed in the context of Eqs. 24 the x- and z-components of velocity in the slot are intimately coupled (Figure 1) in a Hele–Shaw flow. Our asymptotic analysis of the slot flow shows that in dies for which $h \ll L$ and $L \ll W$, the one-dimensional equation for slot flows is entirely valid, even if the slot length L varies. Indeed, such a conclusion was drawn by Durst et al. (1994), who utilized a one-dimensional slot equation and compared results with the analysis of Sartor (1990), who assumed a Hele–Shaw slot flow. One caveat here is that, if the confining walls of the slot are largely sloped (Vrahopoulou, 1991), or if there is a flow obstruction in the slot due to, for example, a lodged object (Vrahopoulou, 1992), additional length scales associated with these complications will arise, and the Hele–Shaw equations will be necessary to analyze their effect on flow.

As discussed in the Introduction, the form of the cavity equation has varied in the literature, and we now focus comments specifically on it. The derived cavity equation, valid for a power-law fluid and given by Eq. 16 with shape-factor definitions, Eqs. 13b and 15b, is *identical* to that utilized by Leonard (1985). This indicates that the inner-cavity equation utilized by Lee and Liu (1989), and later used by Wu et al. (1994), has an inertial term that is incorrect (recall that the equations of Leonard (1985) and Lee and Liu (1989) are identical except for the inertial term). The one-dimensional cavity equations of Sartor (1990) and Durst et al. (1994) have incorrect inertial terms as well, since they are derived by neglecting the x- and y-components of velocity. According to our derivation, these velocity components, which explicitly appear in the differential system, Eq. 3, ultimately contribute to the final form of the cavity equation. The net effect of the neglect of these velocity components is that the coefficient of the inertial terms in the expressions of Sartor (1990) and Durst et al. (1994) should be multiplied by a factor of 2. One other important point regarding the neglect of x- and y-velocity components (Figure 1) in the cavity needs to be made here. Sartor (1990) and Durst et al. (1994) explicitly use the mass-conservation expression, Eq. 31 (note that Sartor utilizes a slightly more general form in his work, consistent with the use of the Hele–Shaw analysis he uses in the slot); yet, it is apparent from our derivation that nonzero x- and y-velocity components are *necessary* to obtain the mass conservation expression, Eq. 31. Thus, the conventional mass-conservation expression is inconsistent with the neglect of x- and y-velocity components in the derivation of the cavity equation itself.

The preceding comments indicate that a predominant difference between Eq. 16 and previous equations is in the form of the inertial term. We now focus on how this term affects flow results, by comparing predictions using Eq. 16 with those obtained using the previously cited alternative forms of the equation. To this end, we apply the cavity equation, Eq. 16, to the case of flow in a duct with a varying cross-sectional area, where the volumetric flow rate, Q, is constant. Furthermore, to isolate the effect of the different proposed inertial terms, we examine a flow regime in which pressure and inertial forces balance in the duct. Under these assumptions, Eq. 16 becomes

$$\rho \frac{1}{A} \frac{d}{dz} \left(\frac{Q^2}{A} \right) = -\frac{dP}{dz}. \tag{36a}$$

In writing Eq. 36a, we have utilized the fact that $\beta = 1$ for plug flow, since for such a case $\bar{V}_z = 1$ in Eq. 15b (note that for the inertial flow considered in this example, we do not need to utilize the viscous trial function form given by Eq. 8). Rearranging Eq. 36a, we obtain

$$\frac{1}{2} \rho Q^2 \frac{d}{dz} \left(\frac{1}{A^2} \right) = -\frac{dP}{dz}, \tag{36b}$$

which is a statement equivalent to Bernoulli's equation, and is often used to describe inertial flow in ducts (Fox and McDonald, 1992). By contrast, under the same assumptions, the cavity equation of Lee and Liu (1989) and Wu et al. (1994) is given by

$$\rho Q^2 \frac{d}{dz} \left(\frac{1}{A^2} \right) = -\frac{dP}{dz}, \tag{37}$$

which indicates that the magnitude of the pressure loss (or recovery, depending on the area variation) across a section of the duct is two times larger than that predicted by Bernoulli's equation, Eq. 36b.

Since Durst et al. (1994) considers the case of a constant-area cavity, the preceding comparison is not appropriate. However, let us consider a comparison of Eq. 16 and that of Durst et al. (1994) and Sartor (1990) for a die cavity of constant area A in which inertial and pressure forces balance, and the flow rate Q varies. Under such circumstances, our Eq. 16 becomes Eq. 36a, which can be written as

$$\rho \frac{1}{A^2} \frac{dQ^2}{dz} = -\frac{dP}{dz}. \tag{38}$$

By contrast, the equations of Sartor (1990) and Durst et al. (1994) become

$$\frac{1}{2} \rho \frac{1}{A^2} \frac{dQ^2}{dz} = -\frac{dP}{dz}. \tag{39}$$

Since $dQ/dz < 0$ for the case of a coating die, Eq. 39 predicts a pressure recovery across the cavity width that is half of that using our equation, Eq. 38; in fact, it is interesting to note that Eq. 39 is identical to Bernoulli's equation. However, in the current application where there is flow loss into the slot of the die, Bernoulli's equation is *not* valid. In the correct equation, Eq. 38, the pressure recovery per unit width is increased compared with Bernoulli's equation since the fluid velocity in the cavity is reduced by the flow leakage into the slot.

We now make some additional comments regarding differences between the derived cavity equation, Eq. 16, and those presented in previous work. In the cavity equations of Sartor (1990) and Durst et al. (1994) viscous stresses in the z (axial) directions were retained. We have demonstrated that such stresses are of order A/W^2 by our asymptotic analysis (i.e., $O(\epsilon^2)$ in Eq. 2). Consequently, for a die that is wide compared with the cavity cross-sectional area, these stresses are negligible over the bulk of the die width. These stresses may still be important near the cavity inlet and end, where the flow appreciably can adjust in the z-direction. In these regions, however, the fully developed flow assumption, used in the determination of the shape factors, will not be valid.

Finally, we comment that the origin of the flow leakage term proposed by Yuan (1995) in the cavity equation is in the second integral on the lefthand side of Eq. 7. As discussed in our derivation (and indicated in Appendix A), the magnitude of this term is likely small relative to the other terms in Eq. 16. Yuan (1995) indicates that the constant parameter associated with his flow leakage term may be adjusted to "fine-tune" the model predictions, presumably after comparison with experiments or three-dimensional computation. Based on our estimate, we propose that if a flow leakage term is indeed used to obtain a reasonable fit of data, the term itself is likely compensating for some other inadequacy of the one-dimensional model, such as, for example, the assumption of fully developed flow and associated shape factors.

Literature Cited

Atkinson, B., M. P. Brocklebank, C. C. Card, and J. M. Smith, "Low Reynolds Number Developing Flows," *AIChE J.*, **15**(4), 548 (1969).

Bird, R. B., R. C. Amstrong, and O. Hassager, *Dynamics of Polymeric Liquids*, Vol. 1, Wiley, New York (1977).

Carley, J. F., "Flow of Melts in Crosshead-Slit Dies; Criteria for Die Design," *J. Appl. Phys.*, **25**(9), 1118 (1954).

Durst, F., U. Lange, and H. Raszillier, "Optimization of Distribution Chambers of Coating Facilities," *Chem. Eng. Sci.*, **49**(2), 161 (1994).

Fox, R. W., and A. T. McDonald, *Introduction to Fluid Mechanics*, Wiley, New York (1992).

Huang, J. C. P., and H. S. Yu, "Pressure Distribution in Porous Ducts of Arbitrary Cross Section," *J. Fluids Eng. Trans. ASME*, **95**, 342 (1973).

Lee, K. Y., and T. J. Liu, "Design and Analysis of a Dual-Cavity Coat-Hanger Die," *Poly. Eng. Sci.*, **29**(15), 1066 (1989).

Leonard, W. K., "Inertia and Gravitational Effects in Extrusion Dies for Non-Newtonian Fluids," *Poly. Eng. Sci.*, **25**(9), 570 (1985).

Liu, T. J., "Fully Developed Flow of Power-Law Fluids in Ducts," *Ind. Eng. Chem. Fundam.*, **22**(2), 183 (1983).

Liu, T. J., C. N. Hong, and K. C. Chen, "Computer-Aided Analysis of Linearly Tapered Coat-Hanger Die," *Poly. Eng. Sci.*, **28**(23), 1517 (1988).

Liu, T. J., and C. N. Hong, "The Pressure Drop/Flow Rate Equation for Non-Newtonian Flow in Channels of Irregular Cross-Section," *Poly. Eng. Sci.*, **28**(23), 1559 (1988).

Lundgren, T. S., E. M. Sparrow, and J. B. Starr, "Pressure Drop Due to the Entrance Region in Ducts of Arbitrary Cross Section," *J. Basic Eng., Trans. ASME, Series D*, **86**(3), 620 (1964).

McKelvey, J. M., and K. Ito, "Uniformity of Flow from Sheeting Dies," *Poly. Eng. Sci.*, **11**(3), 258 (1971).

Miller, C., "Predicting Non-Newtonian Flow Behavior in Ducts Of Unusual Cross Section," *Ind. Eng. Chem. Fundam.*, **11**(4), 524 (1972).

Nguyen, K. T., and M. R. Kamal, "Blow Molding Die Design for Non-axisymmetric Parison Extrusion," *Poly. Eng. Sci.*, **30**(23), 1537 (1990).

Pearson, J. R. A., "Non-Newtonian Flow and Die Design," *Trans. J. Plast. Inst.*, **32**, 239 (1964).

Sartor, L., "Slot Coating: Fluid Mechanics and Die Design," PhD Thesis, Univ. of Minnesota, Minneapolis, (1990).

Schlichting, H., *Boundary Layer Theory*, 7th ed., McGraw-Hill, New York (1979).

Vrahopoulou, E. P., "A Model for Flow in Dies," *Chem. Eng. Sci.*, **46**(2), 629 (1991).

Vrahopoulou, E. P., "Flow Distortions around Particles between Parallel Walls with Application to Streak Formation in Slide-coating Methods," *Chem. Eng. Sci.*, **47**(5), 1027 (1992).

Wang, Y., "The Flow Distribution of Molten Polymers in Slit Dies and Coathanger Dies through Three-dimensional Finite Element Analysis," *Poly. Eng. Sci.*, **31**(3), 204 (1991a).

Wang, Y., "Extrusion of Rubber Compounds and Highly Filled Thermoplastics through Coathanger Dies," *Inter. Poly. Process.*, **4**, 311 (1991b).

Wen, S. H., and T. J. Liu, "Three-dimensional Finite Element Analysis of Polymeric Fluid Flow in an Extrusion Die. Part I: Entrance Effect," *J. Appl. Poly. Sci.*, **34**(10), 827 (1994).

Wu, P. Y., T. J. Liu, and H. M. Chang, "Design of an Extrusion Die with a Variable Choker Bar," *J. Appl. Poly. Sci.*, **51**, 1005 (1994).

Yuan, S. L., "A Flow Model for Non-Newtonian Liquids Inside a Slot Die," *Poly. Eng. Sci.*, **35**(7), 577 (1995).

Appendix A: Estimate of Momentum Leakage into Slot

In this appendix, we estimate the magnitude of the second integral on the lefthand side of Eq. 7. This integral quantifies the leakage of momentum from the cavity into the slot as fluid leaves the cavity and enters the slot, that is, in the vicinity of $x = x_s$ in Figure 2. We begin by assuming some physically reasonable velocity profiles in the vicinity of the slot. We will restrict attention to a Newtonian fluid in the following discussion for simplicity; a shear-thinning fluid would alter the final expression obtained, but not the general conclusion obtained in this section. In the vicinity of the slot, an estimate for V_z is given by

$$V_z \sim -\frac{1}{2\eta_0}\frac{dP}{dz}(hy - y^2), \tag{A1a}$$

where the pressure gradient is that in the cavity, and h is the slot height shown in Figure 2. Equation A1a is merely the expression for pressure-driven fully developed flow in a slot. An estimate for the velocity component V_x is obtained by assuming that the flow is also fully developed across the slot, and thus

$$V_x \sim \frac{6q}{h^3}(hy - y^2). \tag{A1b}$$

In Eq. A1b, $q = Q_o/W$ is the ideal flow per width in the slot; note that this expression is given directly by Eq. 26b for a Newtonian fluid (i.e., $n = 1$ and $m = \eta_0$). Substituting Eq. A1 into the second integral on the left hand side of Eq. 7 and integrating, yields

$$\int_0^h V_x V_z\,|_{x=x_s} dy \sim -\frac{1}{10} Re\, h^2 \frac{dP}{dz}, \tag{A2}$$

where Re is the Reynolds number defined in Eq. 2.

We now examine the order of magnitude of terms in Eq. 7. In particular, let us compare the magnitude of the leakage term (the second integral) on the lefthand side of Eq. 7 with the magnitude of first term on the righthand side of Eq. 7. Using Eq. A2, we can write

$$\frac{\int_0^h V_x V_z|_{x=x_s} dy}{\dfrac{dP}{dz}A} \sim -\frac{1}{10} Re \frac{h^2}{A}. \tag{A3}$$

In a typical die geometry, $h^2 \ll A$, and so, for moderate values of the Reynolds number, Eq. A3 indicates that the leakage term is small; this justifies the neglect of the flow leakage term in Eq. 7. We note that the neglect of the leakage term is equivalent to choosing $V_z = 0$ along the portion of the contour C in Figure 2, where $x = x_s$, that is, where the cavity meets the slot. This justifies the choice of boundary condition Eq. 8b used in obtaining the trial function defined by Eq. 8.

Appendix B: Estimate of Relative Magnitudes of V_z in the Cavity and Slot

In this section, we estimate the relative magnitudes of the velocity components in the cavity and the slot, ultimately leading to Eq. 29. For the purposes of simplicity, we focus attention on a Newtonian fluid; although the final estimate would be different for a generalized Newtonian fluid, the conclusion drawn based on Eq. 29 is valid for such fluids as well. We begin by noting that the average velocity in the cavity is given by

$$V_z|_{\text{cavity}} = \frac{Q}{A}, \tag{B1a}$$

where Q is the local widthwise (z = dependent) volumetric flow rate in the inner cavity, and A is the area of the inner cavity up to the slot, that is, the shaded region in Figure 2 (which can, in general, vary in the z-direction, as indicated in Figure 1). Now we estimate the relationship between the cavity pressure gradient and the volumetric flow rate. To do so, we assume that the viscous forces in the cavity are dominant in the cavity (which is typically the case), and thus from the cavity equation, Eq. 16 (with $m = \eta_0$, $n = 1$), we write:

$$\frac{\eta_0 Q}{KA^2} \sim -\frac{dP}{dz}, \tag{B1b}$$

where K is the viscous shape factor, which is a constant. Then, combining Eqs. B1a and B1b, we obtain the following estimate for the average velocity in the cavity in terms of the cavity pressure gradient:

$$V_z|_{\text{cavity}} \sim -\frac{KA}{\eta_0}\frac{dP}{dz}. \tag{B1c}$$

In the slot, the approximate equation governing the widthwise velocity is given by

$$V_z \sim -\frac{1}{2\eta_0}\frac{dP}{dz}(hy - y^2), \tag{B2a}$$

which is the standard equation for fully developed viscous flow in a slot for a Newtonian fluid (see, for example, Eq. 26b). For such a flow, the height-averaged velocity, $V_z|_{\text{slot}}$, is given by

$$V_z|_{\text{slot}} = \int_0^h V_z dy \sim -\frac{h^2}{12\eta_0}\frac{dP}{dz}. \tag{B2b}$$

Then, combining Eqs. B1c and B2b and rearranging, the result is

$$V_z|_{\text{slot}} \sim \frac{h^2}{12KA} V_z|_{\text{cavity}}. \tag{B3a}$$

Since the expression Eq. B3 is only an order-of-magnitude expression, the proportionality constants 12 and K can be neglected, which leads directly to Eq. 29.

Manuscript received Sept. 13, 1995, and revision received Feb. 27, 1996.